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CMM Advanced Calculus

Terminology

- $!x$ means unique x.
- Odd function $f(-x) = -f(x)$ (Have rotational symmetry)

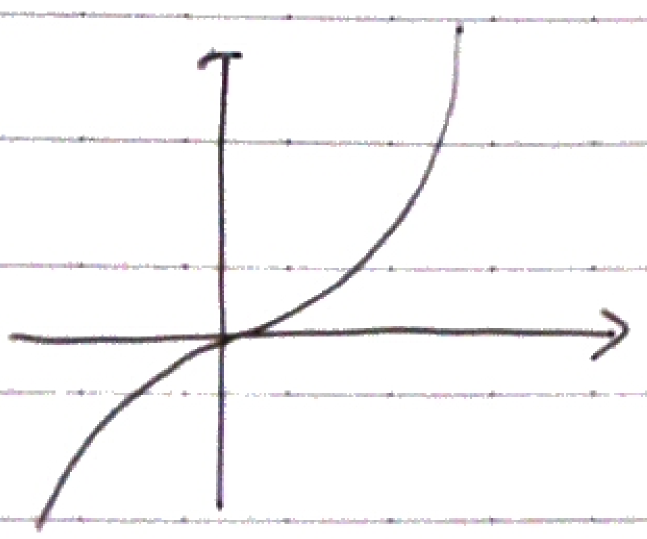
  e.g. $f(x) = x^3$
- Even function $f(x) = f(-x)$ (Symmetrical abt y-axis)
- $\mathbb{R}^n$: Real number in n-dimensional space

Hyperbolic Trigo Functions

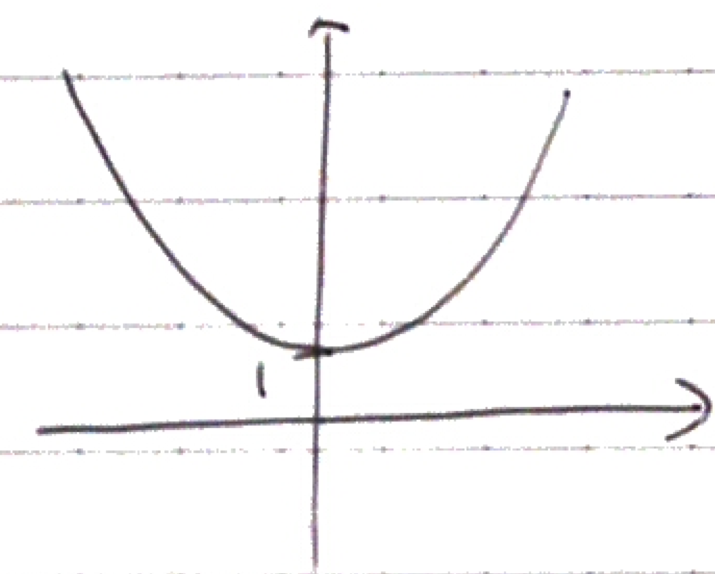
$$\sinh x = \frac{e^x - e^{-x}}{2}$$

$$\cosh x = \frac{e^x + e^{-x}}{2}$$

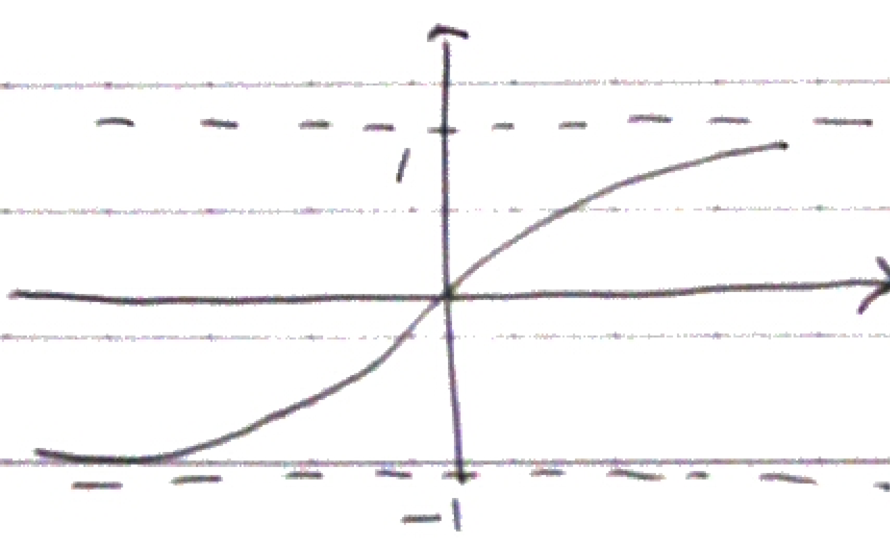
$$\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$



$y = \sinh x$



$y = \cosh x$



$y = \tanh x$

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$\sinh x + \cosh x = e^x$

e^x is a sum of an odd and even function.

Identities

$$\cosh^2 x - \sinh^2 x = 1$$
$$1 - \tanh^2 x = \operatorname{sech}^2 x$$
$$\coth^2 x - 1 = \operatorname{cosech}^2 x$$

Inverse Hyperbolic Function

$$\sinh^{-1} x = \ln \left| x + \sqrt{x^2+1} \right|$$
$$\cosh^{-1} x = \ln \left| x + \sqrt{x^2-1} \right|$$
$$\tanh^{-1} x = \frac{1}{2} \ln \left| \frac{1+x}{1-x} \right|$$

Example:

$$y = \sinh^{-1} x$$
$$x = \sinh y$$
$$= \frac{e^y - e^{-y}}{2}$$
$$2x = e^y - e^{-y}$$
$$e^{2y} - 2xe^y - 1 = 0$$
$$e^y = \frac{-(-2x) \pm \sqrt{4x^2 - 4(1)(-1)}}{2}$$

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$$e^y = \frac{2x \pm \sqrt{4x^2+4}}{2}$$

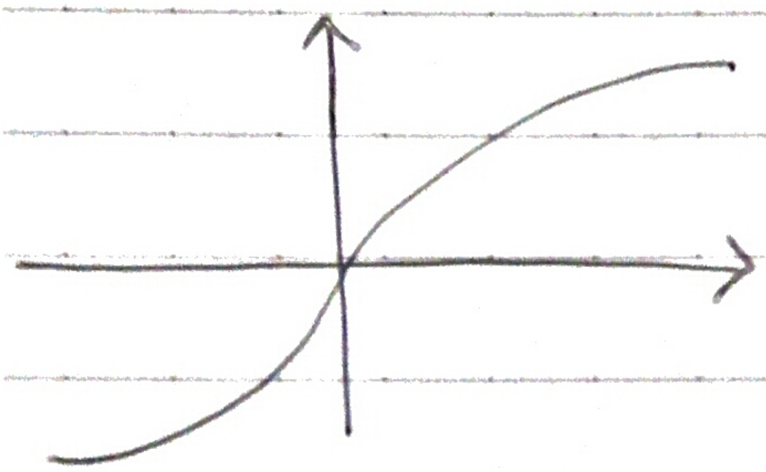
$$= x \pm \sqrt{x^2+1}$$

$$y = \ln\left|x + \sqrt{x^2+1}\right| \quad (\because x < \sqrt{x^2+1})$$

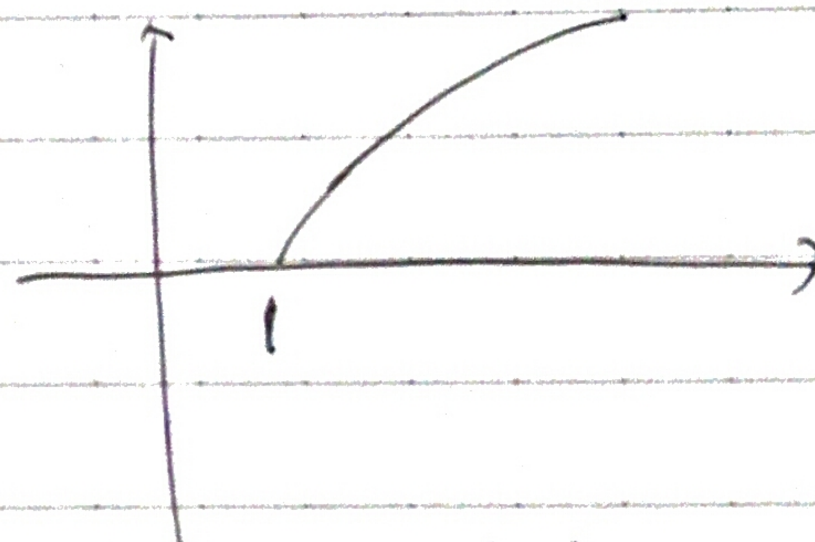
Example 2:

$$y = \tanh^{-1} x$$
$$x = \tanh y$$
$$= \frac{e^y - e^{-y}}{e^y + e^{-y}}$$

$$e^y x + e^{-y} x = e^y - e^{-y}$$
$$e^{2y} x + x - e^{2y} + 1 = 0$$
$$e^{2y}(x-1) + x + 1 = 0$$
$$e^{2y} = \frac{1+x}{1-x}$$
$$y = \frac{1}{2}\ln\left|\frac{1+x}{1-x}\right|$$



$y = \sinh^{-1} x$



$y = \cosh^{-1} x$

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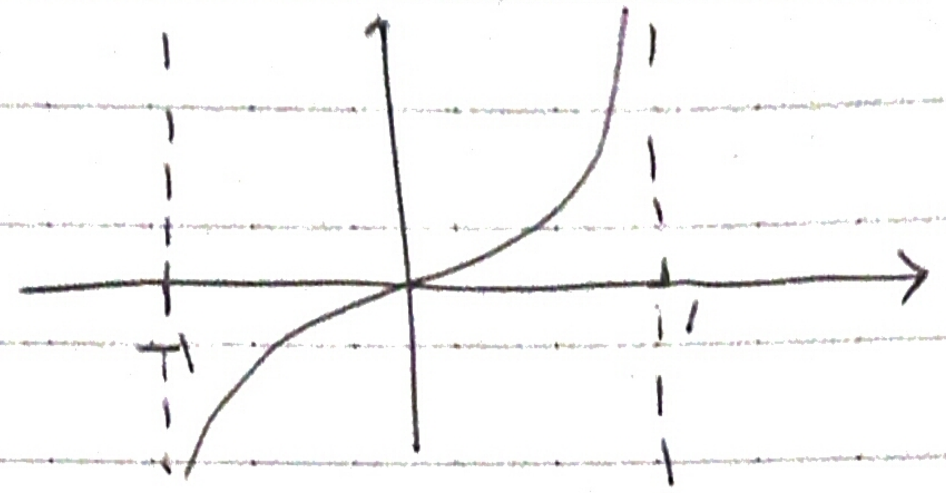


$y = \tanh^{-1} x$

Differentiation of Hyperbolic Functions

$$\frac{d}{dx}(\sinh x) = \cosh x$$
$$\frac{d}{dx}(\cosh x) = \sinh x$$
$$\frac{d}{dx}(\tanh x) = \operatorname{sech}^2 x$$

Vectors

- If vector is beyond 3D, we use $\|\underset{\sim}{v}\|$ to represent magnitude. This is known as vector norm.
- A unit vector is aka a normalized vector
- $\underset{\sim}{u} \cdot \underset{\sim}{v} = |\underset{\sim}{u}||\underset{\sim}{v}| \cos\theta$ (Dot Product)

Cross Product (Vector Product)

- $\underset{\sim}{a} \wedge \underset{\sim}{b} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \underset{\sim}{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \underset{\sim}{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \underset{\sim}{k}$

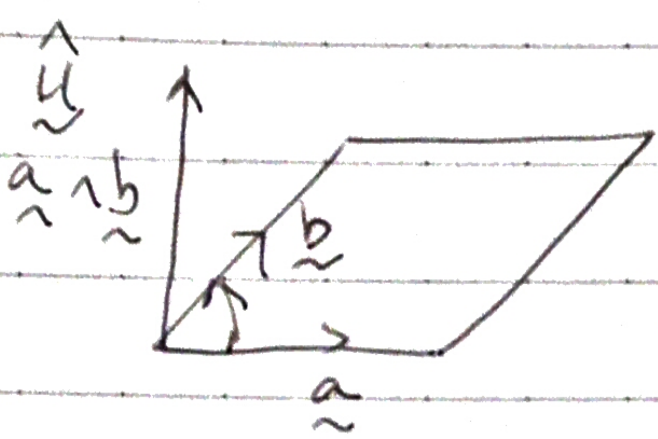
Example: $\underset{\sim}{a} = (2, -1, 6)$ $\quad \underset{\sim}{b} = (-3, 5, 1)$

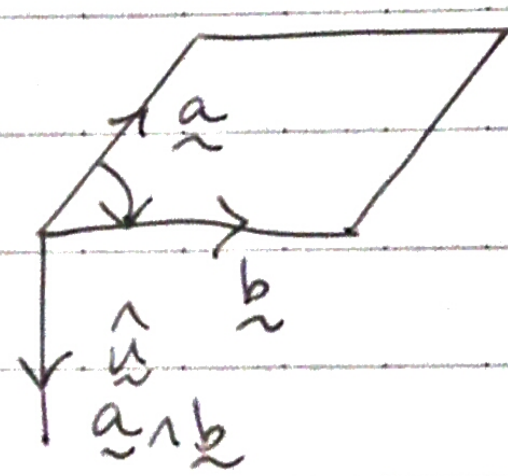
$$\underset{\sim}{a} \wedge \underset{\sim}{b} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ 2 & -1 & 6 \\ -3 & 5 & 1 \end{vmatrix}$$

Note! $\begin{vmatrix} -1 & 6 \\ 5 & 1 \end{vmatrix}$ is known as the <u>minor</u> of $\underset{\sim}{i}$.

$$= -31\underset{\sim}{i} - 20\underset{\sim}{j} + 7\underset{\sim}{k}$$

- $\underset{\sim}{a} \wedge \underset{\sim}{b}$ is orthogonal to both $\underset{\sim}{a}$ and $\underset{\sim}{b}$.
  $\therefore (\underset{\sim}{a} \wedge \underset{\sim}{b}) \cdot \underset{\sim}{a} = 0$ and $(\underset{\sim}{a} \wedge \underset{\sim}{b}) \cdot \underset{\sim}{b} = 0$





- $\underset{\sim}{a} \wedge \underset{\sim}{b} = ab \sin\theta \, \hat{\underset{\sim}{u}}$ (θ goes from $\underset{\sim}{a}$ to $\underset{\sim}{b}$)
- $\underset{\sim}{a} \wedge \underset{\sim}{b} = -\underset{\sim}{b} \wedge \underset{\sim}{a}$

  $\underset{\sim}{a} \wedge \underset{\sim}{a} = \underset{\sim}{0}$ $\quad (\underset{\sim}{a} \wedge k\underset{\sim}{a} = \underset{\sim}{0})$

  $\underset{\sim}{a} \wedge (-\underset{\sim}{a}) = \underset{\sim}{0}$

<u>Triple Scalar Product</u>

- If $\underset{\sim}{a} = (a_1, a_2, a_3)$ $\underset{\sim}{b} = (b_1, b_2, b_3)$ $\underset{\sim}{c} = (c_1, c_2, c_3)$

$$\underset{\sim}{a} \cdot (\underset{\sim}{b} \wedge \underset{\sim}{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

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Determinants (Δ, Δ')

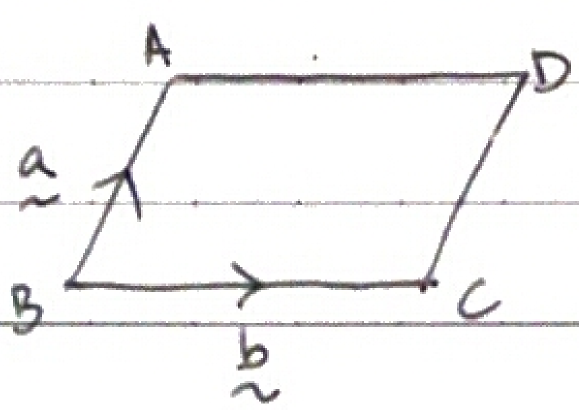
1) If Δ' is obtained from Δ by interchanging a pair of rows or columns, $\Delta = -\Delta'$
2) If Δ has 2 rows (or 2 columns) the same, $\Delta = 0$
3) If all the entries in a row or column have a common factor k, k can be "taken outside" the determinant.
4) If Δ' is obtained from Δ by adding $c \times$ row i to row j (or $c \times$ col i to col j), $\Delta' = \Delta$.

e.g. $\begin{vmatrix} 2 & 4 & 6 \\ 1 & 3 & 9 \\ 5 & 10 & 20 \end{vmatrix} = 10 \begin{vmatrix} 1 & 2 & 3 \\ 1 & 3 & 9 \\ 1 & 2 & 4 \end{vmatrix}$

Triple Vector Product

- $\underset{\sim}{a} \wedge (\underset{\sim}{b} \wedge \underset{\sim}{c}) = (\underset{\sim}{a} \cdot \underset{\sim}{c})\underset{\sim}{b} - (\underset{\sim}{a} \cdot \underset{\sim}{b})\underset{\sim}{c}$

Areas



Area of $\Delta ABC = \frac{1}{2}|\underset{\sim}{a} \wedge \underset{\sim}{b}|$

Area of //ogram $= |\underset{\sim}{a} \wedge \underset{\sim}{b}|$

Example

$A = (-1, -1, 3)$ $B = (2, 4, 1)$, find area of $\triangle OAB$.

$\mathbf{a} = (-1, -1, 3)$ $\mathbf{b} = (2, 4, 1)$

$$|\mathbf{a} \wedge \mathbf{b}| = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & -1 & 3 \\ 2 & 4 & 1 \end{vmatrix}$$

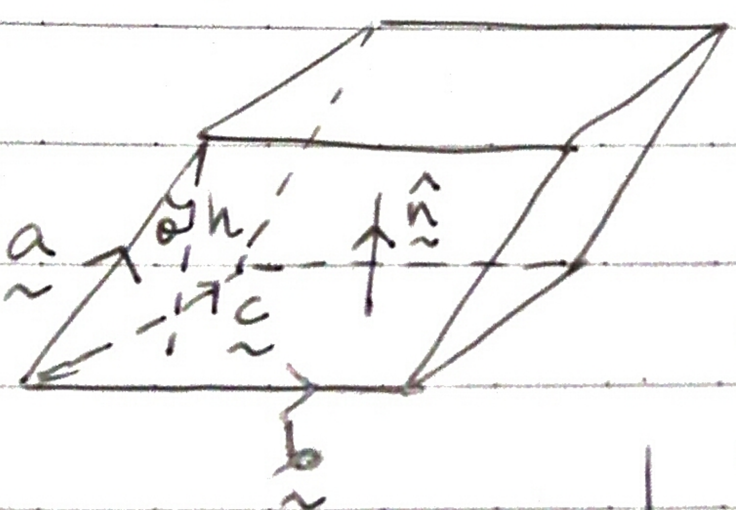
$$= |-13\mathbf{i} + 7\mathbf{j} - 2\mathbf{k}|$$

$$= \sqrt{13^2 + 7^2 + 2^2} = \sqrt{222}$$

Area $= 7.45$ unit^2.

Triple Scalar Product

$\mathbf{a} \cdot (\mathbf{b} \wedge \mathbf{c})$ gives volume of



Vol

$= h \times$ area of base

$= h \times |\mathbf{b} \wedge \mathbf{c}|$

$= (\mathbf{a} \cdot \hat{\mathbf{n}}) \times |\mathbf{b} \wedge \mathbf{c}|$

$= \mathbf{a} \cdot (|\mathbf{b} \wedge \mathbf{c}|\, \hat{\mathbf{n}})$

$= \mathbf{a} \cdot (\mathbf{b} \wedge \mathbf{c})$

$h = |\mathbf{a}| \cos\theta$

$\mathbf{a} \cdot \hat{\mathbf{n}} = |\mathbf{a}||\hat{\mathbf{n}}| \cos\theta$

$= h$