Complex Numbers

Notation

- $z = x + iy \qquad z = (x, y)$
- $(a,b) \pm (c,d) = (a \pm c,\ b \pm d)$

Pf: $(a+ib) \pm (c+id) = (a \pm c) + i(b \pm id)$

- $(a,b).(c,d) = (ac - bd,\ ad + cb)$

Pf: $(a+ib)(c+id) = ac + iad + ibc - bd$

$$= (ac - bd) + i(ad + cb)$$

- $\dfrac{(a,b)}{(c,d)} = \left(\dfrac{ac+bd}{c^2+d^2},\ \dfrac{bc-ad}{c^2+d^2} \right)$

Pf: $$\frac{(a,b)}{(c,d)} = \frac{a+ib}{c+id}$$

$$= \frac{(a+ib)(c-id)}{c^2+d^2}$$

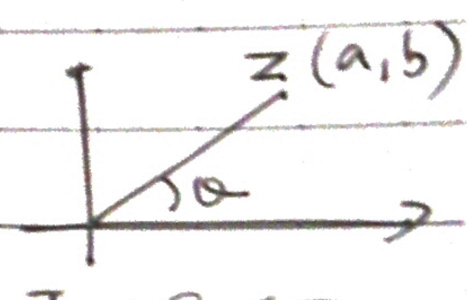
$$= \frac{ac - iad + ibc + bd}{c^2+d^2}$$

$$= \frac{ac+bd}{c^2+d^2} + \frac{bc-ad}{c^2+d^2} i$$

Property

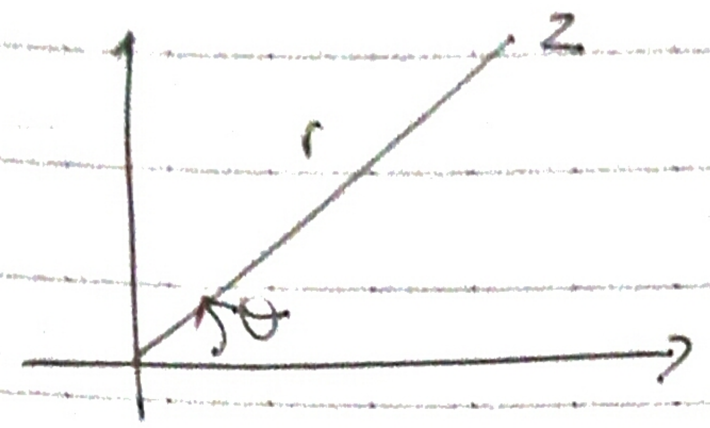
- $\frac{1}{i} = -i$
- $z = (a,b) \qquad |z| = \sqrt{a^2+b^2}$

$\theta = \tan^{-1} \frac{b}{a}$



$-\pi < \theta \leq \pi$

Geometric Representation



$$z = r(\cos\theta + i\sin\theta)$$

$$cis \equiv \cos\theta + i\sin\theta$$

Conjugate

$$\bar{z} = x - iy$$

- $\overline{(\bar{z})} = z$
- $\overline{(z_1 + z_2)} = \bar{z}_1 + \bar{z}_2$
- $\overline{(z_1 z_2)} = \bar{z}_1 \bar{z}_2$
- $z + \bar{z} = 2\,\mathrm{Re}\,z = 2x$
- $z - \bar{z} = (2\,\mathrm{Im}\,z)i = 2iy$

* • $z \cdot \bar{z} = |z|^2$

- $|\bar{z}|^2 = \bar{z}\,\overline{(\bar{z})} = \bar{z} z = |z|^2$
- $\dfrac{z_1}{z_2} = \dfrac{z_1}{z_2}\dfrac{\bar{z}_2}{\bar{z}_2} = \dfrac{z_1 \bar{z}_2}{|z_2|}$

* • $|z_1 z_2|^2 = |z_1||z_2|$

ate

Euler's formula

- $e^{i\theta} = \cos\theta + i\sin\theta$

- when $\theta = \pi$, $\theta = \frac{\pi}{2}$

  $e^{i\pi} = -1$ $\quad e^{i\frac{\pi}{2}} = i$

- $z = r(\cos\theta + i\sin\theta)$ $\quad \therefore \quad z = re^{i\theta}$

  $\bar{z} = r(\cos\theta - i\sin\theta)$

  $= r[\cos(-\theta) + i\sin(-\theta)]$ $\quad \therefore \bar{z} = re^{-i\theta}$

- $z_1 = r_1 e^{i\theta_1}$ $\quad z_2 = r_2 e^{i\theta_2}$

$$\begin{cases} z_1 z_2 = r_1 r_2 e^{i(\theta_1 + \theta_2)} \\ \therefore \ |z_1 z_2| = r_1 r_2 = |z_1||z_2| \\ \arg(z_1 z_2) = \theta_1 + \theta_2 = \arg(z_1) + \arg(z_2) \end{cases}$$

$$\frac{z_1}{z_2} = \frac{r_1 e^{i\theta_1}}{r_2 e^{i\theta_2}} = \frac{r_1}{r_2} e^{i(\theta_1 - \theta_2)}$$

$$\therefore \ \left|\frac{z_1}{z_2}\right| = \frac{r_1}{r_2} = \frac{|z_1|}{|z_2|}$$

$$\arg\left(\frac{z_1}{z_2}\right) = \theta_1 - \theta_2 = \arg(z_1) - \arg(z_2)$$

- $\cos(\theta + 2k\pi) = \cos\theta$

  $\sin(\theta + 2k\pi) = \sin\theta$

  $\therefore e^{i\theta} = e^{i(\theta + 2k\pi)}$

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Generalized Circular and Hyperbolic Functions

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i} \qquad \cos z = \frac{e^{iz} + e^{-iz}}{2}$$

$$\tan z = \frac{\sin z}{\cos z}$$

$$\sinh z = \frac{e^{z} - e^{-z}}{2} \qquad \cosh z = \frac{e^{z} + e^{-z}}{2}$$

$$\tanh z = \frac{\sinh z}{\cosh z}$$

$$\csc z = \frac{1}{\sin z} \qquad \sec z = \frac{1}{\cos z} \qquad \cot z = \frac{1}{\tan z}$$

$$\operatorname{csch} z = \frac{1}{\sinh z} \qquad \operatorname{sech} z = \frac{1}{\cosh z} \qquad \coth z = \frac{1}{\tanh z}$$

$$\sin(iz) = i \sinh z$$

$$\sinh(iz) = i \sin z$$

$$\cos(iz) = \cosh z$$

$$\cosh(iz) = \cos z$$

$$\tan(iz) = i \tanh z$$

$$\tanh(iz) = i \tan z$$

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Example.

Let $z = x + iy$ be any complex number. Find all the values for which $\cosh z = 0$

$$\cosh(a+b) = \cosh a \cosh b + \sinh a \sinh b$$

$$\cosh(x+iy) = \cosh x \cosh iy + \sinh x \sinh iy$$
$$= \cosh x \cos y + i \sinh x \sin y$$

$$\therefore \quad \cosh x \cos y = 0 \quad \text{———} (1)$$
$$\sinh x \sin y = 0 \quad \text{———} (2)$$

From (1)

$\cosh x \neq 0$

$\therefore \cos y = 0$

$y = 2n\pi \pm \frac{\pi}{2} \quad n = 0, 1, 2, \ldots$

From (2)

$$\sin y = \sin\left(2n\pi \pm \frac{\pi}{2}\right)$$
$$= \underbrace{\cos(2n\pi) \quad \text{or} \quad \cos(-2n\pi)}_{\neq 0}$$

$$\therefore \sinh x = 0$$
$$x = 0$$

$$\therefore \quad z = i\left(2n\pi \pm \frac{\pi}{2}\right) \qquad n = 0, 1, 2, \ldots$$

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De Moivres Theorem

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$$
$$(\cos\theta + i\sin\theta)^{-n} = \cos n\theta - i\sin n\theta$$

$$z = e^{i\theta} \qquad \frac{1}{z} = e^{-i\theta} = \bar{z}$$

$$\cos\theta = \frac{1}{2}(z + \bar{z}) = \frac{1}{2}\left(z + \frac{1}{z}\right)$$
$$\therefore \cos n\theta = \frac{1}{2}(z^n + \bar{z}^n) = \frac{1}{2}\left(z^n + \frac{1}{z^n}\right)$$

$$\sin\theta = \frac{1}{2i}(z - \bar{z}) = \frac{1}{2i}\left(z - \frac{1}{z}\right)$$
$$\therefore \sin n\theta = \frac{1}{2i}(z^n - \bar{z}^n) = \frac{1}{2i}\left(z^n - \frac{1}{z^n}\right)$$

Finding roots of complex numbers

Let w be the n^{th} root of a complex number z.

$$\therefore \quad w = z^{\frac{1}{n}}$$
$$= \left[r\left(\cos(\theta + 2k\pi) + i\sin(\theta + 2k\pi)\right)\right]^{\frac{1}{n}}$$
$$= r^{\frac{1}{n}}\left(\cos\frac{\theta + 2k\pi}{n} + i\sin\frac{\theta + 2k\pi}{n}\right)$$
$$k = 0, 1, \ldots, n-1$$

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Example 1:

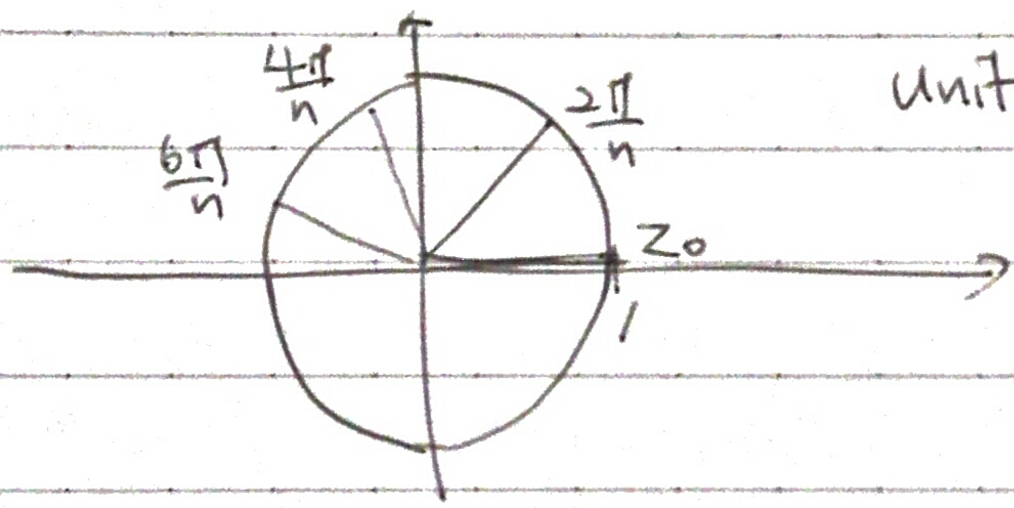
$$z^n = 1$$
$$= 1 + i0 \qquad (r=1, \theta=0)$$
$$= \cos(2k\pi + 0) + i\sin(2\pi k + 0)$$
$$= \cos(2k\pi) + i\sin(2k\pi)$$

$$z = 1^{\frac{1}{n}}$$
$$= [\cos(2k\pi) + i\sin(2k\pi)]^{\frac{1}{n}}$$
$$= \cos\frac{2k}{n}\pi + i\sin\frac{2k}{n}\pi$$

$k = 0, 1, \ldots, n-1$

$$= e^{i\frac{2k}{n}\pi}$$

$\therefore$ nth root of unity $= e^{i\frac{2k\pi}{n}}$ $\quad k = 0, 1, \ldots, n-1$



unit disk

$|z| = 1$

In general, $|z - z_0| = R$ represents a circle centred at z_0 with radius R.

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Example 2:

Calculate the indefinite integral $\int \cos^4\theta \, d\theta$

$$\cos\theta = \frac{1}{2}\left(z + \frac{1}{z}\right)$$

$$\begin{aligned}
\cos^4\theta &= \frac{1}{2^4}\left(z + \frac{1}{z}\right)^4 \\
&= \frac{1}{16}\left(z^4 + 4z^2 + 6 + \frac{4}{z^2} + \frac{1}{z^4}\right) \\
&= \frac{1}{16}\left(z^4 + \frac{1}{z^4}\right) + \frac{1}{4}\left(z^2 + \frac{1}{z^2}\right) + \frac{3}{8} \\
&= \frac{1}{8}\cos 4\theta + \frac{1}{2}\cos 2\theta + \frac{3}{8}
\end{aligned}$$

$$\int \cos^4\theta \, d\theta = \frac{1}{32}\sin 4\theta + \frac{1}{4}\sin 2\theta + \frac{3}{8}\theta + C$$

Example 3:

Express $\cos 4\theta$ in terms of $\cos^n\theta$ and $\sin^n\theta$.

$$\begin{aligned}
\cos 4\theta &= \mathrm{Re}(\cos 4\theta + i\sin 4\theta) \\
&= \mathrm{Re}(\cos\theta + i\sin\theta)^4 \\
&= \mathrm{Re}(c + is)^4 \\
&= \mathrm{Re}[c^4 + 4c^3(is) + 6c^2(is)^2 + \\
&\qquad 4c(is)^3 + (is)^4] \\
&= c^4 - 6c^2s^2 + s^4
\end{aligned}$$

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Example 4:

Calculate $1 + \cos\theta + \cos 2\theta + \dots + \cos n\theta$.

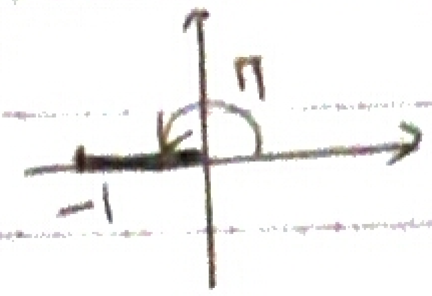
Let $z = e^{i\theta}$

Then $\cos n\theta = Re(e^{i\theta})^n$
$= Re(z^n)$

$$1 + \cos\theta + \cos 2\theta + \dots + \cos n\theta$$
$$= Re\,(z^0 + z^1 + z^2 + \dots + z^n)$$
$$= Re\left(\frac{z^{n+1} - 1}{z - 1}\right)$$
$$= Re\left[\frac{e^{i(n+1)\theta} - 1}{e^{i\theta} - 1}\right]$$
$$= Re\left[\frac{e^{i\frac{n+1}{2}\theta}\left(e^{i\frac{n+1}{2}\theta} - e^{-i\frac{n+1}{2}\theta}\right)}{e^{i\frac{\theta}{2}}\left(e^{i\frac{\theta}{2}} - e^{-i\frac{\theta}{2}}\right)}\right]$$
$$= Re\left[e^{i\frac{n\theta}{2}}\,\frac{z^{\frac{n+1}{2}} - \bar{z}^{\frac{n+1}{2}}}{z^{\frac{\theta}{2}} - \bar{z}^{\frac{\theta}{2}}}\right]$$
$$= Re\left[e^{i\frac{n\theta}{2}}\,\frac{2i\sin\left(\frac{n+1}{2}\right)\theta}{2i\sin\frac{\theta}{2}}\right]$$
$$= \frac{\cos\frac{n\theta}{2}\,\sin\frac{n+1}{2}\theta}{\sin\frac{\theta}{2}}$$

Example 5:

Find the square roots of -1.



$$-1 = \cos(\pi + 2k\pi) + i\sin(\pi + 2k\pi)$$

$$(-1)^{\frac{1}{2}} = \cos\left(\frac{\pi}{2} + k\pi\right) + i\sin\left(\frac{\pi}{2} + k\pi\right) \qquad k = 0, 1$$

$$\therefore \quad (-1)^{\frac{1}{2}} = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} = i$$

$$(-1)^{\frac{1}{2}} = \cos\left(\frac{3\pi}{2}\right) + i\sin\left(\frac{3\pi}{2}\right) = -i$$

Example 6:

Find the fifth roots of $z^5 = -1$.

$$(-1)^{\frac{1}{5}} = \cos\left(\frac{\pi}{5} + \frac{2k\pi}{5}\right) + i\sin\left(\frac{\pi}{5} + \frac{2k\pi}{5}\right) \qquad k = 0, 1, 2, 3, 4$$

$\therefore$ The roots are

$$z_0 = \cos\left(\frac{\pi}{5}\right) + i\sin\left(\frac{\pi}{5}\right)$$

$$z_1 = \cos\left(\frac{3\pi}{5}\right) + i\sin\left(\frac{3\pi}{5}\right)$$

$$z_2 = \cos(\pi) + i\sin(\pi)$$

$$z_3 = \cos\left(\frac{7\pi}{5}\right) + i\sin\left(\frac{7\pi}{5}\right)$$

$$z_4 = \cos\left(\frac{9\pi}{5}\right) + i\sin\left(\frac{9\pi}{5}\right)$$

Example 7:

Find all the cube roots of $z^3 = 1+i$.

$$|1+i| = \sqrt{2}$$

$$\theta = \tan^{-1}\frac{1}{1} = \frac{\pi}{4}$$

$$\therefore 1+i = \sqrt{2}\left[\cos\left(\frac{\pi}{4}+2k\pi\right) + i\sin\left(\frac{\pi}{4}+2k\pi\right)\right]$$

$$= \sqrt{2}\, e^{i\left(\frac{\pi}{4}+2k\pi\right)}$$

$$(1+i)^{\frac{1}{3}} = 2^{\frac{1}{6}}\, e^{i\left(\frac{\pi}{12}+\frac{2k\pi}{3}\right)} \qquad k=0,1,2$$

Propositions

1) $|z_1+z_2| \leq |z_1|+|z_2|$

2) $|w_1-w_2| \geq \big||w_1|-|w_2|\big|$

Singular

$$f(z) = \frac{2z+3}{(z-1)(z-3)}$$

Domain $= \mathbb{C} - \{1,3\}$

We say that $f(z)$ is singular at $z=1,3$.

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Exponential Function

$$e^z = e^{x+iy} = e^x(e^{iy})$$
$$= e^x[\cos y + i\sin y]$$

$$\mathrm{Re}(e^z) = e^x \cos y$$
$$\mathrm{Im}(e^z) = e^x \sin y$$

$$|e^z| = e^x \qquad \text{argument} = y$$

Power Series

$$e^{\pm z} = 1 \pm z + \frac{1}{2!}z^2 \pm \frac{1}{3!}z^3 + \dots$$
$$= \sum_{n=0}^{\infty} (-1)^n \frac{z^n}{n!}$$

$$\sinh z = z + \frac{1}{3!}z^3 + \frac{1}{5!}z^5 + \dots$$
$$= \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!}$$

$$\cosh z = 1 + \frac{1}{2!}z^2 + \frac{1}{4!}z^4 + \dots$$
$$= \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!}$$

$$\sin z = z - \frac{1}{3!}z^3 + \frac{1}{5!}z^5 - \dots$$
$$= \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!}$$

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$$\cos z = 1 - \frac{1}{2!} z^2 + \frac{1}{4!} z^4 - \dots$$
$$= \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!}$$

Argument Function

$$z = x + iy$$
$$= re^{i\theta}$$

$$\arg z = \theta \qquad (-\pi < \theta < \pi)$$
$$\operatorname{Arg} z = \theta + 2k\pi$$

Log Function

If $z = e^w$ and $z = x + iy$, $w = u + iv$

$$w = \log z$$
$$= u + iv$$

$$e^{u+iv} = z = e^u \cdot e^{iv}$$
$$= e^u [\cos v + i \sin v]$$

$$\Rightarrow |z| = e^u$$
$$v = \arg z + 2k\pi$$

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$$\mathrm{Log}\, z = \ln|z| + i\,\mathrm{Arg}\, z$$

$$|z| = e^{u} \Rightarrow u = \ln|z|$$

Log z has infinitely many values $\Rightarrow$ NOT a function.

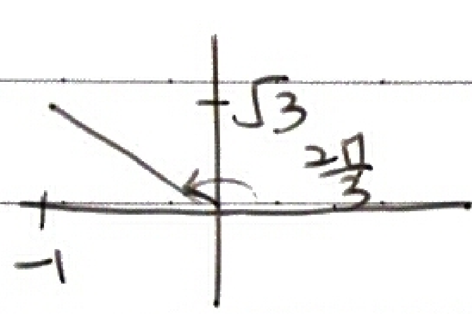
If we take P.v. $(-\pi, \pi)$ of Arg z (i.e. arg z), then the corresponding value of log z is called the P.v. of Log z (i.e. log z)

$$\log z = \ln|z| + i \arg z$$

This is a function with domain $\mathbb{C} - \{0\}$

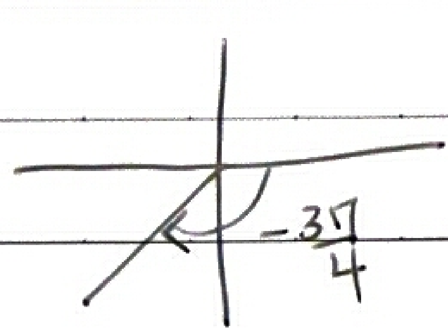
Example:

① $z = -1 + i\sqrt{3}$ $\qquad |z| = \sqrt{1^2 + (\sqrt{3})^2} = 2$



$$\mathrm{Log}\, z = \ln 2 + i\left[\frac{2\pi}{3} + 2k\pi\right]$$
$$k \in \mathbb{Z}$$

② $z = -1 - i$ $\qquad |z| = \sqrt{2}$



$$\mathrm{Log}\, z = \ln\sqrt{2} + i\left[\frac{5\pi}{4} + 2k\pi\right]$$
$$\log z = \ln\sqrt{2} - i\,\frac{3\pi}{4}$$

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(3) $\sin z = 2$

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i} = 2$$

$$e^{iz} - e^{-iz} = 4i$$

$$(e^{iz})^2 - 4ie^{iz} - 1 = 0$$

Let $y = e^{iz}$

$$y^2 - 4iy - 1 = 0$$

$$y = \frac{4i \pm \sqrt{-16 - 4(1)(-1)}}{2}$$

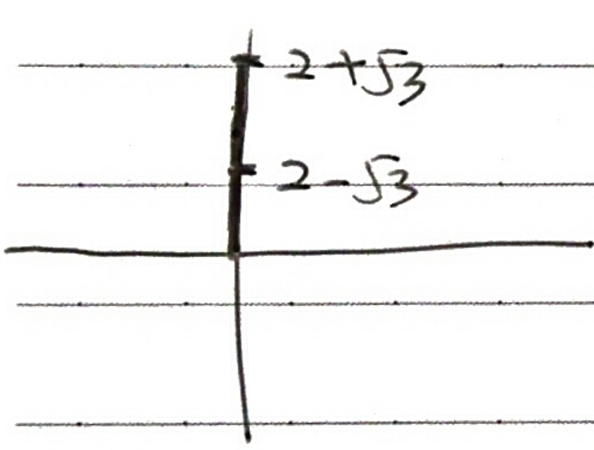
$$= \frac{4i \pm \sqrt{-12}}{2}$$

$$= \frac{4i \pm 2\sqrt{3}i}{2} = 2i \pm \sqrt{3}i$$

$$\therefore e^{iz} = (2 \pm \sqrt{3})i$$

$$iz = \mathrm{Log}\,(2 \pm \sqrt{3})i$$

$$= \ln(2 \pm \sqrt{3}) + i\left(\frac{\pi}{2} + 2k\pi\right)$$



$$(2 - \sqrt{3})(2 + \sqrt{3}) = 1$$

$$\therefore 2 - \sqrt{3} = \frac{1}{2 + \sqrt{3}}$$

$$z = \frac{\ln(2 \pm \sqrt{3})}{i} + \left(\frac{\pi}{2} + 2k\pi\right)$$

$$= -\ln(2 \pm \sqrt{3})i + \left(\frac{\pi}{2} + 2k\pi\right)$$

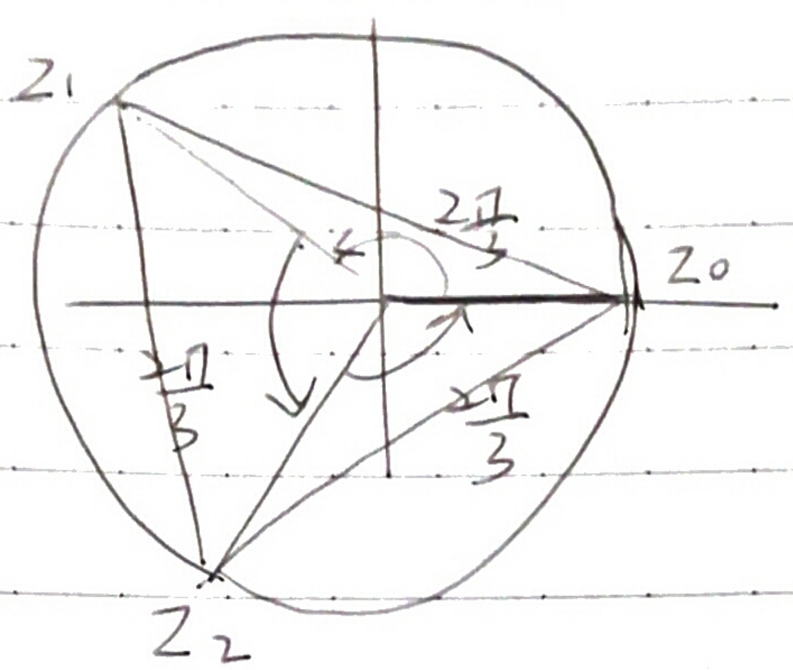
$$\ln(2 - \sqrt{3}) = -\ln(2 + \sqrt{3})$$

$$\therefore z = \left(\frac{\pi}{2} + 2k\pi\right) \pm i\ln(2 + \sqrt{3})$$

Geometric Representation of Roots

nth roots of unity. These roots are equally spaced around the unit circle. $|z|=1$, with $\measuredangle$ between each root $\frac{2\pi}{n}$.

e.g. $z^3 = 1$



$z_0\, z_1\, z_2$ forms an equilateral Δ.

Example:
Find the cartesian equation represented by $|z-3|=4$.

$$|x+iy-3| = 4$$
$$|x-3+iy| = 4$$
$$(x-3)^2 + y^2 = 4^2$$

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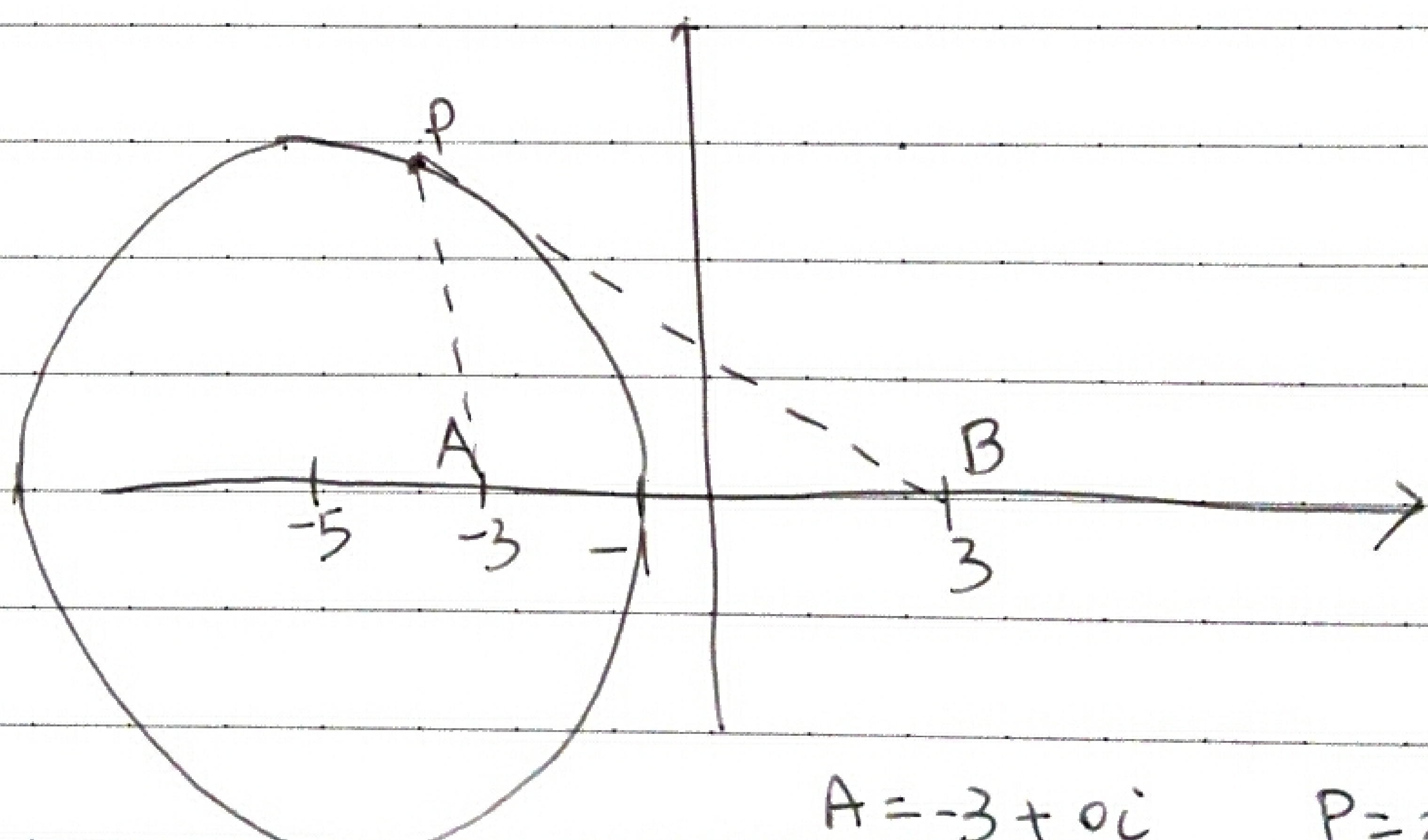
Example:

Find the geometric representation of $\left|\frac{z-3}{z+3}\right| = 2$.

$$|z-3| = 2|z+3|$$
$$|x+iy-3| = 2|x+iy+3|$$
$$(x-3)^2 + y^2 = 4[(x+3)^2 + y^2]$$
$$(x-3)^2 + y^2 = 4(x+3)^2 + 4y^2$$
$$4(x+3)^2 - (x-3)^2 + 3y^2 = 0$$
$$(2x+6-x+3)(2x+6+x-3) + 3y^2 = 0$$
$$(x+9)(3x+3) + 3y^2 = 0$$
$$x^2 + x + 9x + 9 + y^2 = 0$$
$$x^2 + 10x + 9 + y^2 = 0$$
$$(x+5)^2 + y^2 = 4^2$$
$$|z+5| = 4$$



$A = -3 + 0i$ $\quad$ $P = x + iy$

$B = 3 + 0i$

$PA = \sqrt{(x+3)^2 + y^2}$

$PB = \sqrt{(x-3)^2 + y^2}$

$\therefore PB = 2PA$